university of groningen

Group Theory Final Exam

Date: 31 October 2023

Place: Exam Hall 1, Aletta Jacobs Hall

Time: 15:00 – 17:00

Instructions

- Clearly write your name and student number on each page you submit.
- To get full points, you must provide complete arguments and computations. You will get no points if you do not explain your answer.
- While solving a problem, you can use any statement that needs to be proved as a part of another problem even if you did not manage to prove it; e.g. you can use part (a) while solving part (b) even if you did not prove (a). If you get stuck somewhere, consider moving on to the next part and returning to the problem later.
- The examination consists of 5 questions. You can score up to 36 points and you get 4 points for free. This way you will score in total between 4 and 40 points.

Problems

1 Let $\sigma = (1\,2\,3)(2\,4\,6)(1\,2\,6) \in S_6$.

(a) [**2 points**] Determine the order of σ.

(b) [**2 points**] Compute σ^{2023}.

(c) [**2 points**] Is σ conjugate to $(1\,2\,3\,4\,5)$?

(d) [**3 points**] Show that the alternating group A_7 contains an element of order 6.

2 Let G be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of the upper-triangular invertible matrices:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, b, d \in \mathbb{R},\ ad \neq 0 \right\}.$$

You do not have to prove that it is a subgroup.

(a) [**3 points**] Show that the map $f\colon G \to \mathbb{R}^\times \times \mathbb{R}^\times$, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (a, d)$ is a group homomorphism.

(b) [**3 points**] Let

$$U \coloneqq \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\}$$

be the subset of G consisting of upper-triangular matrices with 1's on the diagonal. Show that U is a normal subgroup of G and that there is an isomorphism

$$G/U \cong \mathbb{R}^\times \times \mathbb{R}^\times.$$

3 (a) **[2 points]** Let G be a group and let $H, K \lhd G$ two normal subgroups such that $H \cap K = \{e\}$. Show that $xy = yx$ for all $x \in H$ and $y \in K$.

(b) **[4 points]** Let G be a finite group of order $\#G = p^a q^b$ where $p \neq q$ are primes and $a, b \geq 1$. Assume that G contains a normal subgroup H of order p^a and a normal subgroup K of order q^b. Show that the map $f\colon H \times K \to G$, $(x, y) \mapsto xy$ is an injective homomorphism. Conclude that it is in fact an isomorphism.

(c) **[4 points]** Show that every group of order $2023 = 7 \cdot 17^2$ is abelian.
You can use without proof the fact that every group of order p^2 for a prime p is abelian.

(d) **[2 points]** Find all groups of order 2023 up to isomorphism.

4 **[3 points]** Let H be the subgroup of $\mathbb{Z}^3$ with basis $g_1 = (1, -3, -5)$, $g_2 = (2, 6, 2)$, $g_3 = (0, 0, 4)$. Determine the rank and elementary divisors of $\mathbb{Z}^3/H$.

5 **Prove/Disprove.** For each of the following statements, prove the statement if it is true and disprove it if it is false.

(a) **[2 points]** If a finite group G acts transitively on a set X then all stabilizer subgroups G_x for $x \in X$ have the same size.

(b) **[2 points]** Every group of order 8 is abelian.

(c) **[2 points]** The dihedral group D_5 is simple.

GOOD LUCK! ☺